

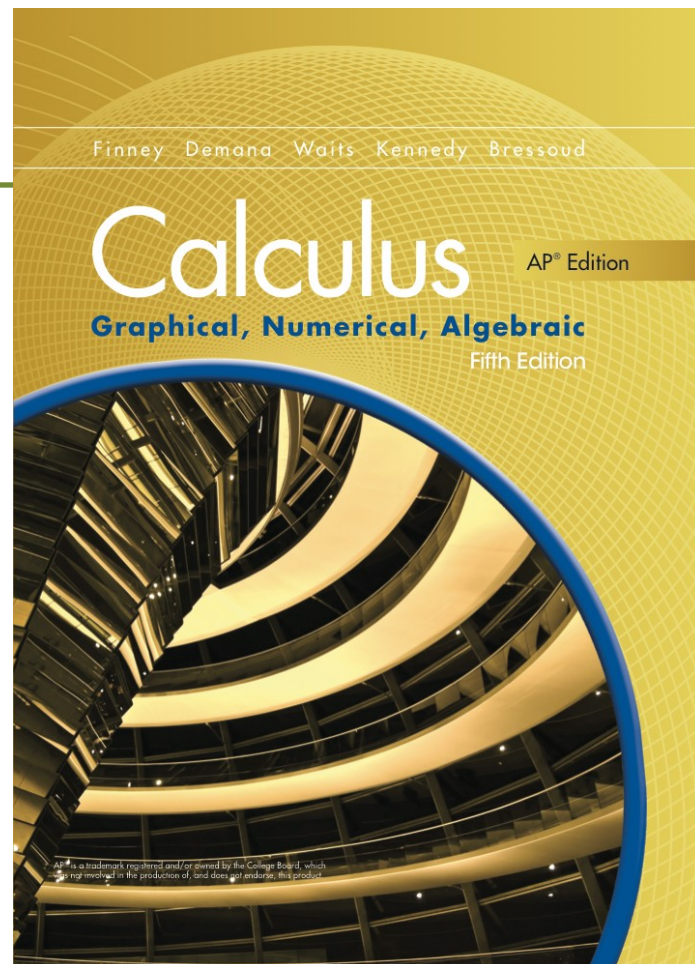
Chapter 7

Differential Equations and Mathematical Modeling



Section 7.2

Antidifferentiation by Substitution



Quick Review

1. Evaluate the definite integral.

$$\int_0^2 x^2 dx$$

Find $\frac{dy}{dx}$ given

2. $y = \int_2^x 2^t dt$

3. $y = (2x^2 + x)^3$

4. $y = \cos(x^2 - 1)$

5. $y = \ln(\tan x)$

Quick Review Solutions

1. Evaluate the definite integral. $\int_0^2 x^2 dx$

$$= \frac{1}{3} x^3 \Big|_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

Find $\frac{dy}{dx}$ given

2. $y = \int_2^x 2^t dt$ $\frac{dy}{dx} = 2^x$

3. $y = (2x^2 + x)^3$ $\frac{dy}{dx} = 3(2x^2 + x)^2 (4x + 1)$

4. $y = \cos(x^2 - 1)$ $\frac{dy}{dx} = -2x \sin(x^2 - 1)$

5. $y = \ln(\tan x)$ $\frac{dy}{dx} = \left(\frac{1}{\tan x} \right) (\sec^2 x)$

$$= \frac{1}{\sin x \cos x}$$

What you'll learn about

- Indefinite integrals
- Properties of indefinite integrals
- Antiderivative formulas arising from known derivatives
- Leibniz notation (differentials) in integrals
- Using substitution to evaluate indefinite integrals
- Using substitution to evaluate definite integrals

... and why

Antidifferentiation techniques were historically crucial for applying the results of calculus.

Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x) dx$.

If F is any function such that $F'(x) = f(x)$, then

$\int f(x) dx = F(x) + C$, where C is an arbitrary constant called the **constant of integration**.

Example Evaluating an Indefinite Integral

Evaluate $\int 2x - \cos x \, dx$.

$$\int 2x - \cos x \, dx = x^2 - \sin x + C$$

Properties of Indefinite Integrals

$$\int kf(x) dx = k \int f(x) dx \text{ for any constant } k$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

Power Functions

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \text{ when } n \neq -1$$

$$\int u^{-1} du = \int \frac{1}{u} du = \ln|u| + C$$

Trigonometric Formulas

$$\int \cos u \, du = \sin u + C$$

$$\int \sin u \, du = -\cos u + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u du = e^u + C$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

$$\int \ln u du = u \ln u - u + C$$

$$\int \log_a u du = \int \frac{\ln u}{\ln a} du = \frac{u \ln u - u}{\ln a} + C$$

Example Paying Attention to the Differential

Let $f(x) = x^2 + 1$ and let $u = x^3$. Find each of the following antiderivatives in terms of x

a. $\int f(x) dx$

b. $\int f(u) du$

c. $\int f(u) dx$

a. $\int f(x) dx = \int (x^2 + 1) dx = \frac{1}{3}x^3 + x + C$

b. $\int f(u) du = \int (u^2 + 1) du = \frac{1}{3}u^3 + u + C = \frac{1}{3}(x^3)^3 + (x^3) + C = \frac{1}{3}x^9 + x^3 + C$

c. $\int f(u) dx = \int (u^2 + 1) dx = \int ((x^3)^2 + 1) dx = \int (x^6 + 1) dx = \frac{1}{7}x^7 + x + C$

Example Using Substitution

Evaluate $\int x^2 e^{x^3} dx$

Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$, from which we conclude that

$\frac{1}{3} du = x^2 dx$. We rewrite the integral and proceed as follows

$$\begin{aligned}\int x^2 e^{x^3} dx &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + C \\ &= \frac{1}{3} e^{x^3} + C\end{aligned}$$

Example Using Substitution

Evaluate $\int 6x\sqrt{1+x^2} dx$

Let $u = 1 + x^2$. Then $du = 2x dx$. Rewrite the integral in terms of u :

$$\int 6x\sqrt{1+x^2} dx = 3 \int \sqrt{u} du$$

$$= 3 \left(\frac{2}{3} u^{\frac{3}{2}} \right) + C$$

$$= 2(1+x^2)^{\frac{3}{2}} + C$$

Example Setting Up a Substitution with a Trigonometric Identity

Evaluate $\int \sin^3 x dx$

$$\begin{aligned}\int \sin^3 x dx &= \int (\sin^2 x) \sin x dx \\&= \int (1 - \cos^2 x) \sin x dx \\&= - \int (1 - u^2) du && \text{let } u = \cos x \text{ and } -du = \sin x dx \\&= -u + \frac{u^3}{3} + C \\&= -\cos x + \frac{\cos^3 x}{3} + C\end{aligned}$$

Example Evaluating a Definite Integral by Substitution

Evaluate $\int_0^2 \frac{x}{x^2 - 9} dx$

Let $u = x^2 - 9$ and $du = 2x dx$. Then $u(0) = 0^2 - 9 = -9$ and $u(2) = 2^2 - 9 = -5$. So,

$$\begin{aligned}\int_0^2 \frac{x}{x^2 - 9} dx &= \frac{1}{2} \int_{-9}^{-5} \frac{du}{u} \\ &= \frac{1}{2} \ln|u|_{-9}^{-5} \\ &= \frac{1}{2} (\ln 5 - \ln 9) \\ &= \frac{1}{2} \ln\left(\frac{5}{9}\right)\end{aligned}$$